Singular Integral Operators in Potential Theory: at the Analysis/Numerical Analysis Interface

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In this project we will attack a long-standing conjecture in potential theory of (Kenig, 1994) regarding the so-called *double-layer potential operator* K on a closed curve $\Gamma \subset \mathbb{R}^2$. The conjecture is that K has essential spectral radius < ½ whenever Γ is the boundary of a bounded Lipschitz domain.

To explain these terms, let $G(x, y) \coloneqq \log(|x - y|)/(2\pi)$, for $x, y \in \mathbb{R}^2$, the so-called *fundamental solution of Laplace's equation*, and suppose $D \subset \mathbb{R}^2$ is a bounded Lipschitz domain with boundary Γ, meaning that, at each $x \in \Gamma$, Γ is locally the graph of some Lipschitz continuous function in some rotated coordinate system. Then the double-layer potential operator K on $L^2(\Gamma, \mathrm{ds})$ (the set of functions on Γ that are square integrable with respect to arc-length measure ds on Γ) is defined by

$$K\phi(x) = \int_{\Gamma} \frac{\partial G(x,y)}{\partial n(y)} \phi(y) \mathrm{ds}(y), \qquad x \in \Gamma,$$

for every $\phi \in L^2(\Gamma, \mathrm{ds})$, where the derivative is in the direction of the unit normal $n(y)$ at y. The spectrum of an operator K is the set of values $\lambda \in \mathbb{C}$ for which $\lambda I - K$ is not invertible, where I is the identity operator. The essential spectrum, $spec_{ess}K$, is arguably the most important part of the spectrum for a bounded operator on a Hilbert space; it is the set of $\lambda \in \mathbb{C}$ for which $\lambda I - K$ is not even approximately invertible in a sense that can be made precise.

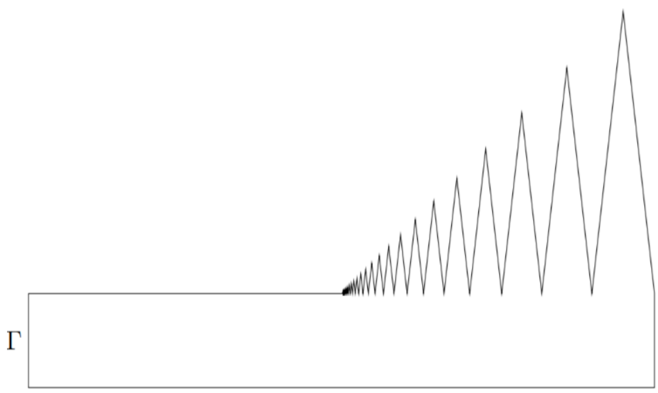

Figure 1. An example of a complicated Lipschitz boundary.

Kenig's conjecture has been shown to hold when Γ is the boundary of a convex domain, and when Γ is a polygon. In this project we will explore whether the conjecture holds for much more complicated boundaries, for example the self-similar boundary Γ shown in Figure 1.

To do this we will use a combination of methods which will depend on the background and interests of the candidate but will include analysis/functional analysis methods and rigorously justified numerical analysis and computation of $spec_{ess}K$ in MATLAB or Python.

In carrying out this work the candidate will work with and learn from the supervisors and will also have opportunities to interact with and visit other leading experts in this area, for example Euan Spence (Bath) and Karl-Mikael Perfekt (Trondheim), and will present their progress at international conferences. For more information about the topic see the recent paper (Chandler-Wilde & Spence 2022).

Chandler-Wilde, S. N., & Spence, E. A. (2022). Coercivity, essential norms, and the Galerkin method for second-kind integral equations on polyhedral and Lipschitz domains. Numerische Mathematik, **150**, 299-371.

Kenig, C. E. (1994). *Harmonic analysis techniques for second order elliptic boundary value problems.* American Mathematical Society.